Malaysian Journal of Mathematical Sciences 4(1): 1-16 (2010)

# Construction of the Green's Function for One Problem of Rectangular Region

**Yusupjon P. Apakov**

*Department of Higher Mathematics, Namangan Engineer-Pedagogical Institute, Namangan, Uzbekistan*

*E-mail: yuapakov@list.ru, apakov.1956@mail.ru*

## ABSTRACT

In this work, the first boundary value problem is studied for the third-order differential equation $U_{xxx} - U_{yy} = f(x, y)$ with multiple characteristics in the region $D = \{(x, y): 0 < x < p,\ 0 < y < l\}$. By the method of energy integral it is proved uniqueness of solution. Green's function is constructed for the first boundary-value problem and through which the explicit solution of the formulated problem is presented.

Keywords: boundary value problem, the third-order differential equation, Green's function, solution.

Subject Classification: 35G15.

## INTRODUCTION

The study of the equation of the 3$^{rd}$ order

$$u_{xxx} - u_{yy} = f\left(x, y\right) \tag{1}$$

with multiple characteristics was originated in papers of H.Block [3], E.Del Vecchio [6,7], L.Cattabriga [5]. Then in the works $[1,11]$, various boundary problems were studied by using the method of potentials.

We have to note that equation (1) in considered conjugated to the equation

$$u_{xxx} + u_{yy} = F\left(x, y\right)$$

which is linear part (in $\nu = 0$) named VT- equation (Viscous Transonic equation),

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}.$$

For $\nu = 1$ expresses VT- equation expresses an axially symmetric flow, and for $\nu = 0$ if expresses a plane-parallel flow and has the form [8,14].

In [9] constructed fundamental solution of the equation (1) using the method of constructing auto modeling solution. These solutions have the following form:

$$U(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} f(t), \quad -\infty < t < \infty,$$
$$V(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} \varphi(t), \quad t<0, \tag{2}$$

where

$$f(t) = \frac{2\sqrt[3]{2}}{\sqrt{3}\pi} t \Psi(\frac{1}{6},\frac{4}{3};\tau),$$

$$\varphi(t) = \frac{36\Gamma(\frac{1}{3})}{\sqrt{3}\pi} t \Phi(\frac{1}{6},\frac{4}{3};\tau), \quad \tau = \frac{4}{27}t^3, \quad t = \frac{x-\xi}{|y-\eta|^{\frac{2}{3}}},$$

$\Psi(a,b;x)$, $\Phi(a,b;x)$ are degenerated hypergeometric functions (see, chapter 6, pp.237, 245,[4]).

Estimates for fundamental solutions were obtained by using estimates for hypergeometric functions when argument of the solution approaches infinity.

For $U(x,y;\xi,\eta)$ the following estimates are valid:

$$\left|\frac{\partial^{h+k}U}{\partial x^h dy^k}\right| \le C_{kh} |y-\eta|^{\frac{1-(-1)^k}{2}} |x-\xi|^{-\frac{1}{2}[2h+3k-1+\frac{3}{2}(1-(-1)^k)]}, \quad \text{as} \quad \left|\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right| \to \infty ,$$

$C_{kh}$ is a constant, $k,h = 0,1,2........$

Similar estimates are true for $V(x,y;\xi,\eta)$ as $(x-\xi)|y-\eta|^{-\frac{2}{3}} \to -\infty$.

In the present paper Green's function for the first boundary-value problem was constructed in a rectangular region and in terms of that the explicit solution of the stated problem was obtained.

## STATEMENT OF THE PROBLEM

In the domain $D=\{(x,y):\ 0<x<p,\ 0<y<l\}$ we consider the equation (1), where $p>0,\ l>0$ are constants.

A function $u(x,y)$ that satisfies the equation (1) in the domain $D$ and belongs to the class $C_{x,y}^{3,2}(D)\cap C_{x,y}^{1,0}(\overline{D})$ is said to be a regular solution of the equation (1).

**Problem** $F_1$. Find the regular solution of the equation (1) in the domain $D$ that satisfies the following boundary-value conditions

$$u(x,0)=\varphi_1(x),\ \ u(x,l)=\varphi_2(x), \tag{3}$$

$$u(0,y)=\psi_1(y),\ \ u(p,y)=\psi_2(y),\ \ u_x(p,y)=\psi_3(y), \tag{4}$$

where

$$\varphi_i(x)\in C[0,p], i=1,2,\ \ \psi_j(y)\in C[0,l],\ \ j=1,2,3, f(x,y)\in C_{x,y}^{0,2}(\overline{D}).$$

Moreover, the following matching conditions

$$\varphi_1(0)=\psi_1(0),\quad \varphi_1(p)=\psi_2(0),\quad \varphi_1'(p)=\psi_3(0),\quad \varphi_2(0)=\psi_1(l),$$
$$\varphi_2(p)=\psi_2(l),\quad \varphi_2'(p)=\psi_3(l),\quad f(x,0)=f(x,l)=0,$$

are satisfied.

Problem $F_1$ for $\varphi_1(x)=\varphi_2(x)=f(x,y)=0$ is considered in [10] and for $\varphi_1(x)=\varphi_2(x)=0$ is considered in [2].

## UNIQUENESS OF THE SOLUTION

With regards to the above, we prove:

**Theorem**. The homogeneous problem $F_1$ has the only trivial solution.

**Proof**. Let the problem $F_1$ has two solutions, say $u_1(x,y)$ and $u_2(x,y)$. Then the function $u(x,y)=u_1(x,y)-u_2(x,y)$ satisfies the equation $u_{xxx}-u_{yy}=0$ and the homogeneous boundary conditions.

Now integrating the identity

$$\frac{\partial}{\partial x}\left(uu_{xx}-\frac{1}{2}u_x^2\right)-\frac{\partial}{\partial y}\left(uu_y\right)+u_y^2=0$$

over the region D and taking into account the boundary conditions we obtain

$$\frac{1}{2}\int_0^l u_x^2(0,y)\,dy+\iint_D u_y^2(x,y)\,dxdy=0.$$

Hence, $u_y(x,y)=0$, i.e. $u(x,y)=\phi(x)$. Equality $u(x,0)=0$, implies $\phi(x)=0$, so $u(x,y)\equiv 0$.

## EXISTENCE OF THE SOLUTION

Consider the following differential operators

$$L\equiv\frac{\partial^3}{\partial\xi^3}-\frac{\partial^2}{\partial\eta^2},\qquad L^*\equiv-\frac{\partial^3}{\partial\xi^3}-\frac{\partial^2}{\partial\eta^2},$$

where $L^*$ is the adjoint operator. Integrating the identity

$$\varphi L[\psi]-\psi L^*[\varphi]\equiv\frac{\partial}{\partial\xi}\left(\varphi\psi_{\xi\xi}-\varphi_\xi\psi_\xi+\varphi_{\xi\xi}\psi\right)-\frac{\partial}{\partial\eta}\left(\varphi\psi_\eta-\varphi_\eta\psi\right)$$

over the region $D$, where $\varphi,\psi$ are sufficiently smooth functions, we obtain

$$\iint_D\left(\varphi L[\psi]-\psi L^*[\varphi]\right)d\xi d\eta=\int_0^l\left(\varphi\psi_{\xi\xi}-\varphi_\xi\psi_\xi+\varphi_{\xi\xi}\psi\right)\Big|_{\xi=0}^{\xi=p}d\eta-\int_0^p\left(\varphi\psi_\eta-\varphi_\eta\psi\right)\Big|_{\eta=0}^{\eta=l}d\xi. \tag{5}$$

Now as the function $\varphi$ we take the fundamental solution of the equation (1), $U(x,y;\xi,\eta)$, that as the function of $(\xi,\eta)$ satisfies the equation

$$L^*[U] \equiv -U_{\xi\xi\xi} - U_{\eta\eta} = 0$$

at $(x,y) \neq (\xi,\eta)$. We take as the function $\psi$ any regular solution $u(x,y)$ of the equation (1). Then identity (5) has the form

$$\iint_D U(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta = \int_0^l \left(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi} u\right)\Bigg|_{\xi=0}^{\xi=p} d\eta - \int_0^p \left(Uu_\eta - U_\eta u\right)\Bigg|_{\eta=0}^{\eta=l} d\xi \tag{6}$$

We prove the following lemma.

**Lemma.** For $\forall \varphi(x) \in C[a;b]$ and $\forall x \neq \xi, y \neq \eta$

$$\lim_{\substack{x \to x_0 \\ \eta \to y}} \int_a^b U_\eta(x,y;\xi,\eta)\varphi(\xi)\, d\xi = -\varphi(x_0).$$

**Proof.** We assume that $y > \eta$ (in the case $y < \eta$ the proof is similar). According to continuity of $\varphi(x)$ at $x_0$ there exists $\delta(\varepsilon)$ such that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.
Using relation (see, [9])

$$U_\eta = -U^* \operatorname{sgn}(y-\eta),$$

where

$$U^*(x,y;\xi,\eta) = \frac{1}{|y-\eta|^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right), \quad f^*(t) = \frac{t}{3\gamma}\Psi\left(\frac{7}{6},\frac{4}{3};\frac{4}{27}t^3\right), \quad \gamma = \frac{3\sqrt{3}\pi}{2^{\frac{1}{3}}},$$

and splitting the interval $[a,b]$ we can represent the integral as the sum of three integrals:

$$\int_a^b U_\eta(x,y;\xi,\eta)\varphi(\xi)d\xi = -\int_a^b U^*(x,y;\xi,\eta)\varphi(\xi)d\xi = -\int_a^b \frac{1}{|y-\eta|^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right)\varphi(\xi)d\xi =$$

$$= -\int_a^{x_1} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)d\xi - \int_{x_1}^{x_2} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)d\xi -$$

$$-\int_{x_2}^{b} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)\, d\xi = I_1 + I_2 + I_3,$$

where

$$x_1 = x_0 - \delta, x_2 = x_0 + \delta.$$

The general term $I_2$ can be rewritten as

$$-\varphi(x_0)\int_{x_1}^{x_2} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)d\xi - \int_{x_1}^{x_2} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\left[\varphi(\xi)-\varphi(x_0)\right]d\xi =$$

$$= I_{21} + I_{22}.$$

The integral $I_{21}$ is evaluated directly by substituting

$$t = \frac{x-\xi}{(y-\eta)^{\frac{2}{3}}},\ \ \xi = x - t(y-\eta)^{\frac{2}{3}},\ \ d\xi = -(y-\eta)^{\frac{2}{3}}\, dt.$$

Indeed,

$$I_{21} = -\varphi(x_0) \int_{\frac{x-x_2}{(y-\eta)^{\frac{2}{3}}}}^{\frac{x-x_1}{(y-\eta)^{\frac{2}{3}}}} f^*(t)dt.$$

If $|x-x_0| < \delta$, then the upper limit is positive and lower limit is negative, and upper limit approaches $+\infty$, lower limit approaches $-\infty$ as $\eta \to y-0$. Therefore in accordance with (see, [9])

$$\int_{-\infty}^{\infty} f^*(t)\, dt = 1$$

we obtain $\lim\limits_{\substack{\eta\to y-0 \\ x\to x_0}} I_{21} = -\varphi(x_0)$.

We show that other integrals $I_{22}, I_1, I_3$ approach zero.

We first estimate the integral $I_{22}$ :

$$\left|I_{22}\right| \leq \int_{x_1}^{x_2} \left| \frac{1}{(y-\eta)^{\frac{2}{3}}} f^* \left( \frac{x-\xi}{(y-\eta)^{\frac{2}{3}}} \right) \right| \left|\varphi(\xi)-\varphi(x_0)\right| d\xi .$$

As $x_1 < \xi < x_2$, then $\left|\xi - x_0\right| < \delta$, so

$$\left|I_{22}\right| \leq \varepsilon \int_{\frac{x-x_2}{(y-\eta)^{\frac{2}{3}}}}^{\frac{x-x_1}{(y-\eta)^{\frac{2}{3}}}} \left|f^*(t)\right| dt .$$

We show that integral

$$\int_{\frac{x-x_2}{(y-\eta)^{\frac{2}{3}}}}^{\frac{x-x_1}{(y-\eta)^{\frac{2}{3}}}} \left|f^*(t)\right| dt ,$$

converges for $x \to x_0$, $\eta \to y-0$. Originally, for $\left|x-x_0\right| < \delta$ upper limit is positive, lower limit is negative and for $\eta \to y-0$ upper limit strives to $+\infty$, lower limit strives to $-\infty$ and we obtain following integral

$$\int_{-\infty}^{+\infty} \left|f^*(t)\right| dt .$$

With regards to the integration domain, one can use the estimate $\left|f^*(t)\right| < C\left|t\right|^{-\frac{5}{2}}$ (see [9]) and for $t \to \infty$, we obtain the $\lim\limits_{\substack{\eta \to y-0 \\ x \to x_0}} I_{22} = 0$.

We have

$$\left|I_1\right| < \left|\int_a^{x_1} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)d\xi\right| < N \int_{\frac{x-x_1}{(y-\eta)^{\frac{2}{3}}}}^{\frac{x-a}{(y-\eta)^{\frac{2}{3}}}} \left|f^*(t)\right|dt \to 0$$

as $x \to x_0$, $\eta \to y-0$ (continuity of the function $\varphi(x)$ on the closed interval implies its boundedness on that interval, i.e. $\left|\varphi(x)\right| \le N$ ).

If $x \to x_0$, then $x - x_1 > 0$ and if $\eta \to y-0$, then the upper and the lower limits approach $+\infty$. Similarly,

$$\left|I_3\right| < \left|\int_{x_2}^{b} \frac{1}{(y-\eta)^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{(y-\eta)^{\frac{2}{3}}}\right)\varphi(\xi)d\xi\right| < N \int_{\frac{x-b}{(y-\eta)^{\frac{2}{3}}}}^{\frac{x-x_2}{(y-\eta)^{\frac{2}{3}}}} \left|f^*(t)\right|dt \to 0$$

as $x \to x_0$, $\eta \to y-0$.

This completes the proof of the Lemma.
According to the above Lemma, (6) implies

$$2u(x,y) = \int_0^l \left(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u\right)\Bigg|_{\xi=0}^{\xi=p} d\eta - \int_0^p \left(Uu_\eta - U_\eta u\right)\Bigg|_{\eta=0}^{\eta=l} d\xi - \iint_D U(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta. \tag{7}$$

Let now $W(x,y;\xi,\eta)$ be any regular solution of the adjoin equation, and $u(x,y)$ be any regular solution of the equation (1). Then letting in (5)

$$\varphi = W(x,y;\xi,\eta), \qquad \psi = u(\xi,\eta)$$

we obtain

$$0 = \int_0^l \left(Wu_{\xi\xi} - W_\xi u_\xi + W_{\xi\xi}u\right)\Bigg|_{\xi=0}^{\xi=p} d\eta - \int_0^p \left(Wu_\eta - W_\eta u\right)\Bigg|_{\eta=0}^{\eta=l} d\xi - \iint_D W(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta. \tag{8}$$

Combining (7) with (8) we obtain

$$2u(x,y)=\int_0^l \left(Gu_{\xi\xi}-G_{\xi}u_{\xi}+G_{\xi\xi}u\right)\Bigg|_{\xi=0}^{\xi=p} d\eta-\int_0^p \left(Gu_{\eta}-G_{\eta}u\right)\Bigg|_{\eta=0}^{\eta=l} d\xi- \tag{9}$$

$$-\iint_D G(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta,$$

where

$$G(x,y;\xi,\eta)=U(x,y;\xi,\eta)-W(x,y;\xi,\eta).$$

Now we construct the function $G(x,y;\xi,\eta)$ having the following properties, at $(x,y)\neq(\xi,\eta)$:

$$\begin{cases} L[G]=0, \\ G(x,0;\xi,\eta)=G(x,l;\xi,\eta)=0, \\ G(0,y;\xi,\eta)=G(p,y;\xi,\eta)=G_x(p,y;\xi,\eta)=0, \end{cases} \tag{10}$$

with respect to the variables $(x,y)$, and

$$\begin{cases} L^*[G]=0, \\ G(x,y;\xi,0)=G(x,y;\xi,l)=0, \\ G(x,y;0,\eta)=G(x,y;p,\eta)=G_{\xi}(x,y;0,\eta)=0 \end{cases} \tag{11}$$

with respect to the variables $(\xi,\eta)$.

To this end we solve the following auxiliary problem:

**Problem** $F_0$. Find the regular solution of the equation (1) satisfying the following boundary-value conditions:

$$u(x,0)=0,\ u(x,l)=0, \qquad 0<x<p, \tag{12}$$

$$u(0,y)=u(p,y)=u_x'(p,y)=0, \qquad 0<y<l. \tag{13}$$

in the domain D.

We seek the solution of the stated problem in the form (see, pp. 95, 211, [15])

$$u(x,y)=\sum_{k=1}^{\infty}X_k(x)\sin\frac{k\pi}{l}y\,. \tag{14}$$

The function $f(x,y)$ can be decomposed over system of eigenfunctions:

$$f(x,y)=\sum_{k=0}^{\infty}f_k(x)\sin\frac{k\pi}{l}y\,, \tag{15}$$

where

$$f_k(x)=\frac{2}{l}\int_0^l f(x,y)\sin\frac{k\pi}{l}ydy\;.$$

Substituting (14) and (15) into (1) we obtain

$$\sum_{k=0}^{\infty}\left(X_k'''(x)+\lambda_k^3X_k(x)-f_k(x)\right)\sin\frac{k\pi}{l}y=0\,.$$

To find the functions $X_k(x)$ we obtain the following problem

$$\begin{cases} L[X_k]\equiv X_k'''(x)+\lambda_k^3X_k(x)=f_k(x)\\ X_k(0)=X_k(p)=X_k'(p)=0, \end{cases} \tag{16}$$

where

$$\lambda_k^{\,3}=\left(\frac{k\pi}{l}\right)^2.$$

We seek the solution of the problem (16) by the method of construction of Green's function $G_k(x,\xi)$ (see, p.96, [12]) that possesses the following properties:

1. $G_k(x,\xi)$ is continuous and has continuous derivative with respect to $x$ in $0\le x\le p$.
2. Its derivative of the second order with respect to $x$ at $x=\xi$ has a jump discontinuity, with the jump being equal to 1, i.e.

$$\left.\frac{\partial^2G_k(x,\xi)}{\partial x^2}\right|_{x=\xi+0}-\left.\frac{\partial^2G_k(x,\xi)}{\partial x^2}\right|_{x=\xi-0}=1\,.$$

3. In each of intervals $0 \le x < \xi$ and $\xi < x \le p$ the function $G_k(x,\xi)$ as a function of $x$ satisfies the equation

$$L[G_k] = \frac{\partial^3 G_k}{\partial x^3} + \lambda_k^3 G_k = 0 .$$

4. $G_k(0,\xi) = G_k(p,\xi) = G_{kx}(p,\xi) = 0$.

The following function is the Green function for the problem (16):

$$\begin{aligned} G_k(x,\xi) = \frac{1}{\bar{\Delta}} \Bigg\{ & 2e^{-\lambda_k\left(\frac{3}{2}p+x-\xi\right)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right) - 2e^{-\frac{\lambda_k}{2}(2x+\xi)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right) \\ & -2e^{-\lambda_k\left(\frac{3}{2}p-\xi-\frac{x}{2}\right)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k (p-x) + \frac{\pi}{6}\right] + \\ & +2e^{-\frac{\lambda_k}{2}(\xi-x)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k (\xi-x) + \frac{\pi}{6}\right] + \\ & +4e^{-\frac{\lambda_k}{2}(3p+\xi-x)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k (p-\xi)\right] \sin\frac{\sqrt{3}}{2}\lambda_k x \Bigg\}, \qquad 0 \le x \le \xi, \end{aligned} \tag{17}$$

$$\begin{aligned} G_k(x,\xi) = \frac{1}{\bar{\Delta}} \Bigg\{ & -2e^{-\frac{\lambda_k}{2}(2x+\xi)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right) - 2e^{-\lambda_k\left(\frac{3}{2}p-\xi-\frac{x}{2}\right)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k (p-x) + \frac{\pi}{6}\right] + \\ & + e^{-\lambda_k(x-\xi)} + 4e^{-\frac{\lambda_k}{2}(3p+\xi-x)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k (p-x) + \frac{\pi}{6}\right] \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right) \Bigg\}, \\ & \xi \le x \le p, \end{aligned}$$

where

$$\bar{\Delta} = 3\lambda_k^2 \left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right) \ne 0 .$$

The function defined by (17) can easily be verified to possess all the properties formulated in definition of Green's function for the problem (16). Hence, the solution of the problem (16) has the form

$$X_k(x) = \int_0^p G_k(x,\xi) f_k(\xi) d\xi . \tag{18}$$

Then combining (14) with (18) we obtain the solution of the problem $F_0$:

$$u(x,y)=\sum_{k=1}^{\infty}\int_0^p G_k(x,\xi)f_k(\xi)d\xi\sin\frac{\pi k}{l}y=\int_0^p\sum_{k=1}^{\infty}G_k(x,\xi)\sin\frac{\pi ky}{l}f_k(\xi)d\xi. \quad (19)$$

If the series (19) and the partial derivatives $u_{xxx}$, $u_{yy}$ converge uniformly in $D=\{(x,y):0<x<p,0<y<l\}$, then the function $u(x,y)$ is the solution of the problem $F_0$.

We estimate the function (19):

$$|u(x,y)|\le\left|\int_0^p\sum_{k=1}^{\infty}G_k(x,\xi)\sin\frac{\pi ky}{l}f_k(\xi)d\xi\right|\le\int_0^p\sum_{k=1}^{\infty}|G_k(x,\xi)|\left|\sin\frac{\pi k}{l}y\right||f_k(\xi)|d\xi\le$$
$$\le\int_0^p\sum_{k=1}^{\infty}|G_k(x,\xi)||f_k(\xi)|d\xi. \quad (20)$$

Under the conditions stated above regarding the function $f(x,y)$, the following inequality is true (see, p. 55, [13])

$$|f_k(\xi)|\le\frac{M_1}{k^2},\ \ M_1=const>0,$$

since $f_n(\xi)$ are coefficients of Fourier in decomposition of $f(x,y)$ in $(0,l)$. Therefore from (20) we obtain

$$|u(x,y)|\le\int_0^p\sum_{k=1}^{\infty}|G_k(x,\xi)||f_k(\xi)|d\xi\le M_2\int_0^p\sum_{k=1}^{\infty}\frac{1}{k^2}|G_k(x,\xi)|d\xi. \quad (21)$$

Evaluating the estimators of the function $G_k(x,\xi)$ we obtain from (17)

$$|G_k(x,\xi)|\le\begin{cases}\dfrac{10}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2}+\dfrac{2}{3}\dfrac{e^{-\frac{1}{2}\lambda_k\delta_1}}{\lambda_k^2}, & 0\le x<\xi, \quad 0<\delta_1<\xi-x,\\[2ex] \dfrac{8}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2}+\dfrac{1}{3}\dfrac{e^{-\frac{1}{2}\lambda_k\delta_2}}{\lambda_k^2}, & \xi<x\le l, \quad 0<\delta_2<x-\xi\end{cases},$$

or

$$|G_k(x,\xi)|\le\frac{10}{3}\frac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2}+\frac{2}{3}\frac{e^{-\frac{1}{2}\lambda_k\delta}}{\lambda_k^2}=M_3k^{-\frac{4}{3}}. \quad (22)$$

Then from (21) we take $\left|u(x,y)\right| \le M_4 k^{-\frac{10}{3}}$, and series (19) is uniformly converged.

Hence, the series under integral in (20) converges uniformly.
We show that the series of derivatives $u_{xxx}$ converges uniformly. As

$$\left|\frac{\partial^3}{\partial x^3}u(x,y)\right| \le \int_0^p \sum_{k=1}^{\infty}\left|\frac{\partial^3}{\partial x^3}G_k(x,\xi)\right|\left|f_k(\xi)\right|d\xi \le M_1\int_0^p \sum_{k=1}^{\infty}\frac{1}{k^2}\left|\lambda_k^3 G_k(x,\xi)\right|d\xi, \quad (23)$$

then

$$\left|\lambda_k^3 G_k(x,\xi)\right| \le \frac{10}{3}\lambda_k e^{-\frac{3}{2}\lambda_k p} + \frac{2}{3}\lambda_k e^{-\frac{1}{2}\lambda_k \delta} \le M_6 k^{\frac{2}{3}}.$$

Then from (23) we have

$$\left|\frac{\partial^3}{\partial x^3}u(x,y)\right| \le M_7 k^{-\frac{4}{3}}, \ \ M_i = const > 0,\ i = \overline{1,7},$$

and series (23) is uniformly converged.

Since $\frac{\partial^2}{\partial y^2}u(x,y) = \frac{\partial^3}{\partial x^3}u(x,y)$ ,the same reasoning applies to the series associated with the derivative $u_{yy}$ .This implies that it is possible to differentiate the series (19) term by term, which is necessary condition for $u(x,y)$ to satisfy the equation (1). Changing the order of summation and integration is always valid, since the series under integral (19) is convergent with respect to $\xi$ .

By replacing $f_n(\xi)$ with their values in the solution (20), we obtain the final solution of auxiliary problem $F_0$ in the form

$$u(x,y) = \int_0^p \sum_{k=1}^{\infty} G_k(x,\xi)\sin\frac{\pi k y}{l} f_k(\xi)d\xi =$$

$$= \frac{2}{l}\int_0^p \sum_{k=1}^{\infty} G_k(x,\xi)\int_0^l f(\xi,\eta)\sin\frac{\pi k}{l}\eta \sin\frac{\pi k}{l} y d\eta d\xi =$$

$$=\int_0^p\int_0^l f(\xi,\eta)\frac{2}{l}\sum_{k=1}^{\infty}G_k(x,\xi)\sin\frac{\pi k}{l}\eta\sin\frac{\pi k}{l}y\,d\xi d\eta=\int_0^p\int_0^l G(x,\xi,y,\eta)f(\xi,\eta)\,d\xi\,d\eta,$$

where

$$G(x,\xi,y,\eta)=\frac{2}{l}\sum_{k=1}^{\infty}G_k(x,\xi)\,\sin\frac{\pi k}{l}\eta\,\sin\frac{\pi k}{l}y. \qquad (24)$$

A trivial verification shows that the function $G(x,\xi,y,\eta)$ satisfies all the conditions of the problems (10) and (11). The function (24) is Green's function of the first boundary-value problem for the region D. Convergence of the series (24) follows from the estimate (22) for the function $G_k(x,\xi)$ at $x\neq\xi$.

Now taking into account the boundary conditions (10),(11) for the function $G(x,\xi,y,\eta)$ and the boundary conditions (3),(4) for $u(x,y)$ from (9) we obtain the solution of $F_1$ in explicit form:

$$2u(x,y)=\int_0^l G_{\xi\xi}(x,y,p,\eta)\psi_2(\eta)\,d\eta-\int_0^l G_{\xi\xi}(x,y,0,\eta)\psi_1(\eta)\,d\eta-$$
$$-\int_0^l G_{\xi}(x,y,p,\eta)\psi_3(\eta)\,d\eta+\int_0^p G_{\eta}(x,y,\xi,l)\varphi_2(\xi)\,d\xi-$$
$$-\int_0^p G_{\eta}(x,y,\xi,0)\varphi_1(\xi)\,d\xi-\iint_D G(x,y,\xi,\eta)f(\xi,\eta)\,d\xi d\eta.$$

Eventually, we gained the solution of $F_1$ problem expressed as an integral equation in other works as precise appearance.

## REFERENCES


[1] Abdinazarov, C. 1989. On one third-order equation. *Izv. AN UzSSR ser. fiz.-mat. Nauk*, **6**: 3-6. (in Russian).

[2] Apakov, Yu. P. 2007. On the solution of one boundary-value problem for a non homogenous equation of the 3rd order. *Thesis of the International Conference "Differential Equations, Theory of*

*Functions and Applications"* 28 May – 2 June 2007. Novosibirsk, 65-66. (in Russian).

[3] Block, H. 1912-1913. Sur les equations lineaires aux derivees partielles a carateristiques multiples. Note 1, Ark. Mat. Astron. Fys. 7, No.**13** (1912), 1-34; Note 2, ibid. 7, No. **21** (1912), 1-30; Note 3, ibid. 8, No.**23** (1912-1913), 1-51.

[4] Beitmen G., Erdeyi. 1973. A Higher transcendent functions V. 1, *Nauka, Moscow*, 296 (in Russian).

[5] Cattabriga, L. 1961. Potenziali di linea e di dominio per equazioni non paraboliche in due variabili a caratteristiche multiple. *Rendiconti del seminario matimatico della univ.di Padova*, **31**:1-45.

[6] Del Vecchio E. 1915. Sulle equazioni $Z_{xxx} - Z_y + \varphi_1(x, y) = 0$, $Z_{xxx} - Z_{yy} + \varphi_2(x, y) = 0$. *Memorie R. Accad. Sci. Torino.Ser.2*, **66**:1-41.

[7] Del Vecchio E., 1916. Sur deux problemes d'integration pour les equations paraboliques $Z_{\xi\xi\xi} - Z_\eta = 0$, $Z_{\xi\xi\xi} - Z_{\eta\eta} = 0$. *Arkiv for Mat. Astr. och Fys.* Bd.11.

[8] Diesperov, V.N. 1972. On the Green function for the linearized viscous transonic equation. *Jurnal vychislitelnoy matematiki I matematicheskoy fiziki.*, **12**(5): 1265-1279. (in Russian).

[9] Djuraev, T.Dj., Apakov, Yu. P. 2007. On the automodel solution of one third-order equation with multiple characteristics. *Herald of the Samara State Technical University, series of Physical and Mathematical Sciences*, **2**(15):18-26. (in Russian).

[10] Irgashev, Yu., Apakov, Yu. P. 2006. The first boundary value problem for the third-order equation of pseudoelliptic type. *Uzbek Math. Journal*, **2**: 44-51. (in Russian).

[11] Khashimov, A.R. 1995. On one problem for the equation of the mixed type with multiple characteristics. *Uzbek Math. Journal,* **2**: 93-97. (in Russian).

[12] Krasnov, M.L., Kiselyev, A.I., Makarenko, G.I. 1976. Integral Equations. *The 2nd edition Moscow, Nauka*, **216**. (in Russian).

[13] Kudryavtsev, L.D., 1989. *Kurs Matematicheskogo Analiza (Calculus)*. Vol. **3**. The 2nd edition. Moscow. «Visshaya Shkola». 352. (in Russian).

[14] Ryzhov, O.S., 1952. Asymptotical portrait for flow of rotation bodies with the stream of viscous and head conducting gas. *Prikladnaya matematika I mekhanika*. **2**(6): 1004-1014. (in Russian).

[15] Tikhonov, A.N., Samarskiy, A.A. 1977. Equation of Mathematical Physics. *Moskov, "Nauka"*. 735. (in Russian).